

BACKWARD ORBIT CONJECTURE FOR LATTÈS MAPS

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ABSTRACT. For a Lattès map $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ defined over a number field K , we prove a conjecture on the integrality of points in the backward orbit of $P \in \mathbb{P}^1(\overline{K})$ under ϕ .

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1. INTRODUCTION

Let $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a rational map of degree ≥ 2 defined over a number field K , and write ϕ^n for the n th iterate of ϕ . For a point $P \in \mathbb{P}^1$, let $\phi^+(P) = \{P, \phi(P), \phi^2(P), \dots\}$ be the *forward orbit* of P under ϕ , and let

$$\phi^-(P) = \bigcup_{n \geq 0} \phi^{-n}(P)$$

be the *backward orbit* of P under ϕ . We say P is ϕ -*preperiodic* if and only if $\phi^+(P)$ is finite.

Viewing the projective line \mathbb{P}^1 as $\mathbb{A}^1 \cup \{\infty\}$ and taking $P \in \mathbb{A}^1(K)$, a theorem of Silverman [Sil93] states that if ∞ is not a fixed point for ϕ^2 , then $\phi^+(P)$ contains at most finitely many points in \mathcal{O}_K , the ring of algebraic integers in K . If S is the set of all archimedean places for K , then \mathcal{O}_K is the set of points in $\mathbb{P}^1(K)$ which are S -integral relative to ∞ (see section 2). Replacing ∞ with any point $Q \in \mathbb{P}^1(K)$ and S with any finite set of places containing all the archimedean places, Silverman's Theorem can be stated as: If Q is not a fixed point for ϕ^2 , then $\phi^+(P)$ contains at most finitely many points which are S -integral relative to Q .

A conjecture for finiteness of integral points in backward orbits was stated in [Soo11, Conj. 1.2].

Conjecture 1.1. *If $Q \in \mathbb{P}^1(\overline{K})$ is not ϕ -preperiodic, then $\phi^-(P)$ contains at most finitely many points in $\mathbb{P}^1(\overline{K})$ which are S -integral relative to Q .*

In [Soo11], Conjecture 1.1 was shown true for the powering map $\phi(z) = z^d$ with degree $d \geq 2$, and consequently for Chebyshev polynomials. A generalized version of this conjecture, which is stated over a dynamical family of maps $[\varphi]$, is given in [GI13, Sec. 4]. Along those lines, our goal is to prove a general form of Conjecture 1.1 where $[\varphi]$ is the family of Lattès maps associated to a fixed elliptic curve E defined over K (see Section 3).

2. THE CHORDAL METRIC AND INTEGRALITY

2.1. The Chordal Metric on \mathbb{P}^N . Let M_K be the set of places on K normalized so that the product formula holds: for all $\alpha \in K^*$,

$$\prod_{v \in M_K} |\alpha|_v = 1.$$

For points $P = [x_0 : x_1 : \cdots : x_N]$ and $Q = [y_0 : y_1 : \cdots : y_N]$ in $\mathbb{P}^N(\overline{K}_v)$, define the v -adic chordal metric as

$$\Delta_v(P, Q) = \frac{\max_{i,j} (|x_i y_j - x_j y_i|_v)}{\max_i (|x_i|_v) \cdot \max_i (|y_i|_v)}.$$

Note that Δ_v is independent of choice of projective coordinates for P and Q , and $0 \leq \Delta_v(\cdot, \cdot) \leq 1$ (see [KS09]).

2.2. Integrality on Projective Curves. Let C be an irreducible curve in \mathbb{P}^N defined over K and S a finite subset of M_K which includes all the archimedean places. A *divisor* on C defined over \overline{K} is a finite formal sum $\sum n_i Q_i$ with $n_i \in \mathbb{Z}$ and $Q_i \in C(\overline{K})$. The divisor is *effective* if $n_i > 0$ for each i , and its *support* is the set $\text{Supp}(D) = \{Q_1, \dots, Q_\ell\}$.

Let $\lambda_{Q,v}(P) = -\log \Delta_v(P, Q)$ and $\lambda_{D,v}(P) = \sum n_i \lambda_{Q_i,v}(P)$ when $D = \sum n_i Q_i$. This makes $\lambda_{D,v}$ an arithmetic distance function on C (see [Sil87]) and as with any arithmetic distance function, we may use it to classify the integral points on C .

For an effective divisor $D = \sum n_i Q_i$ on C defined over \overline{K} , we say $P \in C(\overline{K})$ is S -integral relative to D , or P is a (D, S) -integral point, if and only if $\lambda_{Q_i^\sigma, v}(P^\tau) = 0$ for all embeddings $\sigma, \tau : \overline{K} \rightarrow \overline{K}$ and for all places $v \notin S$. Furthermore, we say the set $\mathcal{R} \subset C(\overline{K})$ is S -integral relative to D if and only if each point in \mathcal{R} is S -integral relative to D .

As an example, let C be the projective line $\mathbb{A}^1 \cup \{\infty\}$, S be the archimedean place of $K = \mathbb{Q}$, and $D = \infty$. For $P = x/y$, with x and y relatively prime in \mathbb{Z} , we have $\lambda_{D,v}(P) = -\log |y|_v$ for each prime v . Therefore, P is S -integral relative to D if and only if $y = \pm 1$; that is, P is S -integral relative to D if and only if $P \in \mathbb{Z}$.

From the definition we find that if $S_1 \subset S_2$ are finite subsets of M_K which contains all the archimedean places, then P is a (D, S_2) -integral point implies that P is a (D, S_1) -integral point. Similarly, if $\text{Supp}(D_1) \subset \text{Supp}(D_2)$, then P is a (D_2, S) -integral point implies that P is also a (D_1, S) -integral point. Therefore enlarging S or $\text{Supp}(D)$ only enlarges the set of (D, S) -integrals points on $C(\overline{K})$.

For $\phi : C_1 \rightarrow C_2$, a finite morphism between projective curves and $P \in C_2$, write

$$\phi^* P = \sum_{Q \in \phi^{-1}(P)} e_\phi(Q) \cdot Q$$

where $e_\phi(Q) \geq 1$ is the ramification index of ϕ at Q . Furthermore, if $D = \sum n_i Q_i$ is a divisor on C , then we define $\phi^* D = \sum n_i \phi^* Q_i$.

Theorem 2.1 (Distribution Relation). *Let $\phi : C_1 \rightarrow C_2$ be a finite morphism between irreducibly smooth curves in $\mathbb{P}^N(\overline{K})$. Then for $Q \in C_1$, there is a finite set of places S , depending only on ϕ and containing all the archimedean places, such that $\lambda_{P,v} \circ \phi = \lambda_{\phi^*P,v}$ for all $v \notin S$.*

Proof. See [Sil87, Prop. 6.2b] and note that for projective varieties the $\lambda_{\delta W \times V}$ term is not required, and that the big-O constant is an M_K -bounded constant not depending on P and Q . \square

Corollary 2.2. *Let $\phi : C_1 \rightarrow C_2$ be a finite morphism between irreducibly smooth curves in $\mathbb{P}^N(\overline{K})$, let $P \in C_1(\overline{K})$, and let D be an effective divisor on C_2 defined over K . Then there is a finite set of places S , depending only on ϕ and containing all the archimedean places, such that $\phi(P)$ is S -integral relative to D if and only if P is S -integral relative to ϕ^*D .*

Proof. Extend S so that the conclusion of Theorem 2.1 holds. Then for $D = \sum n_i Q_i$ with each $n_i > 0$ and $Q_i \in C_2(\overline{K})$, we have that

$$\lambda_{\phi^*D,v}(P) = \lambda_{D,v}(\phi(P)) = \sum n_i \lambda_{Q_i,v}(\phi(P)).$$

So $\lambda_{\phi^*D,v}(P) = 0$ if and only if $\lambda_{Q_i,v}(\phi(P)) = 0$. \square

3. MAIN RESULT

Let E be an elliptic curve, $\psi : E \rightarrow E$ a morphism, and $\pi : E \rightarrow \mathbb{P}^1$ be a finite covering. A *Lattès map* is a rational map $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ making the following diagram commute:

$$\begin{array}{ccc} E & \xrightarrow{\psi} & E \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{P}^1 & \xrightarrow{\phi} & \mathbb{P}^1 \end{array}$$

For instance, if E is defined by the Weierstrass equation $y^2 = x^3 + ax^2 + bx + c$, $\psi = [2]$ is the multiplication-by-2 endomorphism on E , and $\pi(x, y) = x$, then

$$\phi(x) = \frac{x^4 - 2bx^2 - 8cx + b^2 - 4ac}{4x^3 + 4ax^2 + 4bx + 4c}.$$

Fix an elliptic curve E defined over a number field K , and for $P \in \mathbb{P}^1(\overline{K})$ define:

$$[\varphi] = \left\{ \phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \left| \begin{array}{l} \text{there exist } K\text{-morphism } \psi : E \rightarrow E \\ \text{and finite covering } \pi : E \rightarrow \mathbb{P}^1 \text{ such} \\ \text{that } \pi \circ \psi = \phi \circ \pi \end{array} \right. \right\}$$

$$\Gamma_0 = \bigcup_{\phi \in [\varphi]} \phi^+(P)$$

$$\Gamma = \left(\bigcup_{\phi \in [\varphi]} \phi^-(\Gamma_0) \right) \cup \mathbb{P}^1(\overline{K})_{[\varphi]\text{-preper}}$$

A point Q is $[\varphi]$ -preperiodic if and only if Q is ϕ -preperiodic for some $\phi \in [\varphi]$. We write $\mathbb{P}^1(\overline{K})_{[\varphi]\text{-preper}}$ for the set of $[\varphi]$ -preperiodic points in $\mathbb{P}^1(\overline{K})$.

Theorem 3.1. *If $Q \in \mathbb{P}^1(\overline{K})$ is not $[\varphi]$ -periodic, then Γ contains at most finitely many points in $\mathbb{P}^1(\overline{K})$ which are S -integral relative to Q .*

Proof. Let Γ'_0 be the $\text{End}(E)$ -submodule of $E(\overline{K})$ that is finitely generated by the points in $\pi^{-1}(P)$, and let

$$\Gamma' = \{ \xi \in E(\overline{K}) \mid \lambda(\xi) \in \Gamma'_0 \text{ for some non-zero } \lambda \in \text{End}(E) \}.$$

Then $\pi^{-1}(\Gamma) \subset \Gamma'$. Indeed, if $\pi(\xi) \in \Gamma$ is not $[\varphi]$ -preperiodic, then ξ is non torsion and $(\phi_1 \circ \pi)(\xi) \in \Gamma_0$ for some Lattès map ϕ_1 . So $(\pi \circ \psi_1)(\xi) \in \Gamma_0$ for some morphism $\psi_1 : E \rightarrow E$, and this gives $(\pi \circ \psi_1)(\xi) = \phi_2(P)$ for some Lattès map ϕ_2 . Therefore $\psi_1(\xi) \in (\pi^{-1} \circ \phi_2)(P) = (\psi_2 \circ \pi^{-1})(P)$ for some morphism $\psi_2 : E \rightarrow E$. Since any morphism $\psi : E \rightarrow E$ is of the form $\psi(X) = \alpha(X) + T$ where $\alpha \in \text{End}(E)$ and $T \in E_{\text{tors}}$ (see [Sil07, 6.19]), we find that there is a $\lambda \in \text{End}(E)$ such that $\lambda(\xi)$ is in Γ'_0 , the $\text{End}(E)$ -submodule generated by $\pi^{-1}(P)$. Otherwise, if $\pi(\xi) \in \Gamma$ is $[\varphi]$ -preperiodic, then $\pi(E(\overline{K})_{\text{tors}}) = \mathbb{P}^1(\overline{K})_{[\varphi]\text{-preper}}$ ([Sil07, Prop. 6.44]) gives that ξ may be a torsion point; again $\xi \in \Gamma'$ since $E(\overline{K})_{\text{tors}} \subset \Gamma'$. Hence $\pi^{-1}(\Gamma) \subset \Gamma'$.

Let D be an effective divisor whose support lies entirely in $\pi^{-1}(Q)$, let \mathcal{R}_Q be the set of points in Γ which are S -integral relative to Q , and let \mathcal{R}'_D be the set of points in Γ' which are S -integral relative to D . Extending S so that Theorem 2.1 holds for the map $\pi : E \rightarrow \mathbb{P}^1$, and since $\text{Supp}(D) \subset \text{Supp}(\pi^*Q)$, we have: if $\gamma \in \Gamma$ is S -integral relative to Q , then $\pi^{-1}(\gamma)$ is S -integral relative to D . Therefore $\pi^{-1}(\mathcal{R}_Q) \subset \mathcal{R}'_D$. Now π is a finite map and $\pi(E(\overline{K})) = \mathbb{P}^1(\overline{K})$; so to complete the proof, it suffices to show that D can be chosen so that \mathcal{R}'_D is finite.

From [Sil07, Prop. 6.37], we find that if Λ is a nontrivial subgroup of $\text{Aut}(E)$, then $E/\Lambda \cong \mathbb{P}^1$ and the map $\pi : E \rightarrow \mathbb{P}^1$ can be determined explicitly. The four possibilities for π , which are $\pi(x, y) = x, x^2, x^3$, or y correspond respectively to the four possibilities for Λ , which are $\Lambda = \mu_2, \mu_4, \mu_6$,

or μ_3 , which in turn depends only on the j -invariant of E . (Here, μ_N denotes the N th roots of unity in \mathbb{C} .)

First assume that $\pi(x, y) \neq y$. Since Q is not $[\varphi]$ -preperiodic, take $\xi \in \pi^{-1}(Q)$ to be non-torsion. Then $-\xi \in \pi^{-1}(Q)$ since $\Lambda = \mu_2, \mu_4$, or μ_6 , and $\xi - (-\xi) = 2\xi$ is non-torsion. Taking $D = (\xi) + (-\xi)$, [GI13, Thm. 3.9(i)] gives that \mathcal{R}'_D is finite.

Suppose that $\pi(x, y) = y$. Then $\pi^{-1}(Q) = \{\xi, \xi', \xi''\}$ where $\xi + \xi' + \xi'' = 0$ and ξ is non-torsion since Q is not $[\varphi]$ -preperiodic. Assuming that both $\xi - \xi'$ and $\xi - \xi''$ are torsion give that 3ξ is torsion, and this contradicts the fact that ξ is non-torsion. Therefore, we may assume that $\xi - \xi'$ is non-torsion. Now taking $D = (\xi) + (\xi')$, [GI13, Thm. 3.9(i)] again gives that \mathcal{R}'_D is finite. Hence \mathcal{R}_Q , the set of points in Γ which are S -integral relative to Q , is finite. \square

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